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Pattern Recognition Letters 20 (1999) 347–354

Pattern Recognition Letters

Estimation of fractal dimension of images using a fixed mass approach

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Received 12 May 1998; received in revised form 4 December 1998

Abstract

A method from the field of chaotic dynamics is applied for the estimation of fractal dimension of images. The method is compared with other well-known algorithms on a set of computer generated images of known fractal dimension. The results confirm the superiority of the method in terms of accuracy, dynamic range and computational time. © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: Fractal dimension; Box-counting method; Correlation algorithm; *k*th nearest neighbour method

1. Introduction

Fractal geometry has received much attention as a useful tool for image analysis. The intensity surface of an image can be considered as a fractal object whose properties are quantified numerically by the use of the fractal dimension. For an image, the fractal dimension is a non-integer number between 2 and 3 and it is a measure of the roughness of its intensity surface. Experiments have demonstrated that the fractal dimension is highly correlated with the human perception of image texture; the rougher the texture appears the larger is the fractal dimension.

Several methods have been proposed for the estimation of the fractal dimension of images. The most widely used method is box-counting, which is based on the covering of the intensity surface with cubes of fixed size. The fractal dimension is obtained by the scaling of the number of non-empty cubes with the size of the cubes. However, this method underestimates the true fractal dimension for relatively high values (e.g. above 2.6), mainly due to the discretisation of the image domain and the quantisation of the grey levels. Keller et al. (1989) proposed a modification of the box-counting method based on linear interpolation slightly improving its performance. However, this modified method still underestimates the true fractal dimension for very high values (e.g. above 2.8), while it increases the computational time. The correlation algorithm (Theiler, 1990) provides a very elegant formulation for estimating fractal dimension. According to this algorithm the dimension is obtained by the scaling of the mass of spheres (or boxes) with the size of the spheres.

In this paper, an alternative method, from the field of chaotic dynamics, is proposed for the

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PII: S0167-8655(99)00004-5

estimation of the fractal dimension of an image. The method is called the kth nearest neighbour method and it has been previously applied for the estimation of the fractal dimension of strange attractors, resulting in very accurate estimates (Termonia and Alexandrowicz, 1983). Additionally, the method is relatively simple, fast and has a very wide dynamic range regarding the number of points at the log–log plot used for the estimation of the fractal dimension.

In Section 2, the box-counting algorithm, its modifications and the correlation algorithm are described. The proposed method is introduced and implemented in Section 3. The proposed method is then compared with the other methods on computer generated images of known fractal dimension in Section 4. In Section 5, the choice of the parameters of the new method and its time complexity are discussed.

2. Fractal dimension and methods for estimation

There are several fractal dimensions such as the Hausdorff dimension (D_H), the Minkowski–Bouligand dimension (D_M), the box-counting dimension (D_B), the entropy dimension (D_E), etc. (for details, see (Maragos and Sun, 1993)). These dimensions are more or less capable of quantifying the degree of fragmentation of curves and surfaces. The above dimensions satisfy the following relation:

$$D_H \leqslant D_M = D_B = D_E.$$

The box-counting dimension is used widely, mainly due to the relative case with which it is estimated.

The fractal (box-counting) dimension, FD, of an image is obtained by the scaling:

$$N(r) = cr^{-\mathrm{FD}} \quad \text{for } r \to 0, \tag{1}$$

where $N(r)$ denotes the number of cubes of size r, needed to cover the intensity surface of the image and c is a constant. In practice, only discrete data are available and therefore the limit $r \to 0$ cannot be reached. In order to overcome this difficulty, the fractal dimension is obtained by the slope of the best fitting line at the points $(-\log r,\ \log N(r))$, for various values of r.

Several approaches have been proposed for the calculation of $N(r)$. Voss (1988) proposed an elegant way to carry out this calculation. Specifically, $N(r)$ can be obtained by

$$N(r) = \sum_{m=1}^{M} mP(m, r), \tag{2}$$

where M is the number of pixels of the image and $P(m, r)$ denotes the probability that there are m points within a cube of size r, centred about an arbitrary point of the image. The probability $P(m, r)$, can be estimated by the following relation:

$$P(m, r) = \frac{n(m, r)}{N_{\mathrm{ref}}},$$

where N_{ref} is the number of randomly chosen reference points from the image and $n(m, r)$ denotes the number of cubes of size r, centred around each reference point, containing m points of the image.

Sarkar and Chaudhuri (1994) proposed a different way for calculating $N(r)$. Specifically, their approach considers an image of size $M \times M$. The domain of image is partitioned into grids of size $r \times r$. On each grid there is a column of boxes of size $r \times r \times h$, where h is the height of a single box. If the total number of grey levels is G then $[G/h] = [M/r]$. The boxes are numbered sequentially $1, 2, \ldots$ Let the minimum and maximum grey level of image in (i, j)th grid fall in box number p and q, respectively. Then $n_r(i, j) = q - p + 1$ is the contribution of the (i, j)th grid in $N(r)$. Taking contributions from all grids, we have

$$N(r) = \sum_{i,j} n_r(i, j).$$

Because of the differential nature of computing $n_r(i, j)$ the method is called the differential box-counting (DBC) approach. The authors claimed that calculating $N(r)$ in this manner gives a better approximation to the boxes intersecting the image intensity surface, especially when there are sharp grey level variations in neighbouring pixels.

A modification of the DBC, called the relative differential box-counting (RDBC) method, was proposed by Jin et al. (1995). According to this method, $N(r)$ is obtained by the following equation:

$$N(r) = \sum_{i,j} \mathrm{ceil}(kd_r(i,j)/r),$$

where $d_r(i,j)$ denotes the difference between the maximum and the minimum grey level of the image in the grid (i,j), $k = M/G$ and ceil(x) stands for the ceiling of x (the smallest integer $\geqslant x$).

A very popular way to compute dimension is to use the correlation algorithm, which estimates dimension based on the statistics of pairwise distances. According to this algorithm the (correlation) dimension is defined as (Theiler, 1990)

$$v = \lim_{r \to 0} \frac{\log C(r)}{\log r},$$

where $C(r)$ is the correlation integral given by

$$C(r) = \frac{\text{\# of distances less than } r}{\text{\# of distances altogether}}.$$

The correlation algorithm provides a particularly elegant formulation and simultaneously has the substantial advantage that the function $C(r)$ is approximated even for r as small as the minimum interpoint distance. For an image with N pixels, $C(N,r)$ has a dynamic range of $O(N^2)$. Logarithmically speaking, this range is twice that available in the box-counting method.

3. The *k*th nearest neighbour method

Theiler (1990) argued that the box-counting method belongs to a class of algorithms based on fixed size (all the cubes have the same size, r) and as such is not well suited for the estimation of the fractal dimension. Instead, the so called fixed-mass methods can provide a better estimation. According to these methods, the scaling of the sizes of cubes so that they contain the same number of points (mass), is considered. The main representative of this class of algorithms is the kth nearest neighbour method. An early implementation of the method, in the context of chaotic dynamics, was developed by Termonia and Alexandrowicz (1983). They suggested the following scaling of the average distance, $\langle r_k \rangle$, of a point to its kth nearest neighbour as a function of k:

$$\langle r_k \rangle \sim k^{1/\mathrm{FD}}. \tag{3}$$

Grassberger (1985) corrected Eq. (3) as follows:

$$\langle r_k^\gamma \rangle = G(k,\gamma) k^{\gamma/D(\gamma)}, \tag{4}$$

where $\gamma = (1-q)D_q$, $D(\gamma) = D_q$ and $G(k,\gamma)$ is a function of k and γ, which is near unity for large k. D_q is the multifractal dimension (Theiler, 1990) of order q and for $q = 0$, the fractal dimension is obtained, that is $\mathrm{FD} = D_0$. Also for $q = 0$ it follows that $\gamma = D(\gamma) = \mathrm{FD}$ which means that the fractal dimension is the fixed point of the function $D(\gamma)$.

In Fig. 1, the plot of $\log\langle r_k^\gamma \rangle$ versus $\log k$, with $\gamma = 2.5$ and $G(k,\gamma) = 1$, is shown for a fractal image of size 256×256 with 256 grey levels, generated by the Random Midpoint Displacement (RMD) method (Saupe, 1988, p. 100). The plot clearly confirms the scaling given by Eq. (4).

In our implementation, the fractal dimension of an image is estimated iteratively, using Eq. (4), for $k = k_{\min}, \ldots, k_{\max}$ (k integer), as follows:

Step 1. N_{ref} reference points from the image, $\{X_m\}$ ($m = 1, 2, \ldots, N_{\mathrm{ref}}$), are chosen randomly.

Step 2. An initial value of γ, γ_0, is chosen arbitrarily and $G(k,\gamma_0)$ is set to unity for every k. Since the fractal dimension of an image is between 2 and 3, it would be better to choose γ_0 in this range, e.g. $\gamma_0 = 2.5$. However, as will be shown in Section 5, the performance of the method is not affected significantly by the initial value of γ.

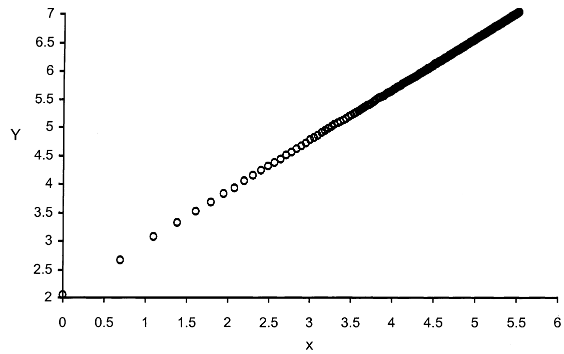

Fig. 1. Plot of $y = \log\langle r_k^\gamma \rangle$ versus $x = \log k$ for an image, with $\gamma = 2.5$.

Step 3. Let $s_0 = \lceil\sqrt{1+k_{\max}}\rceil$ ($\lceil a\rceil$ denotes the smallest integer $\geqslant$ a). Then for each reference point $\boldsymbol{X}_m$, the number of points, $N(m,s)$, within the cube of size $s = s_0$, centred around $\boldsymbol{X}_m$, is computed. The size s is increased by one and $N(m,s)$ is updated, until $N(m,s) \geqslant k_{\max}$. Then the distances, from $\boldsymbol{X}_m$, of the points lying within the cube of size s, are sorted in ascending order. From the sorted

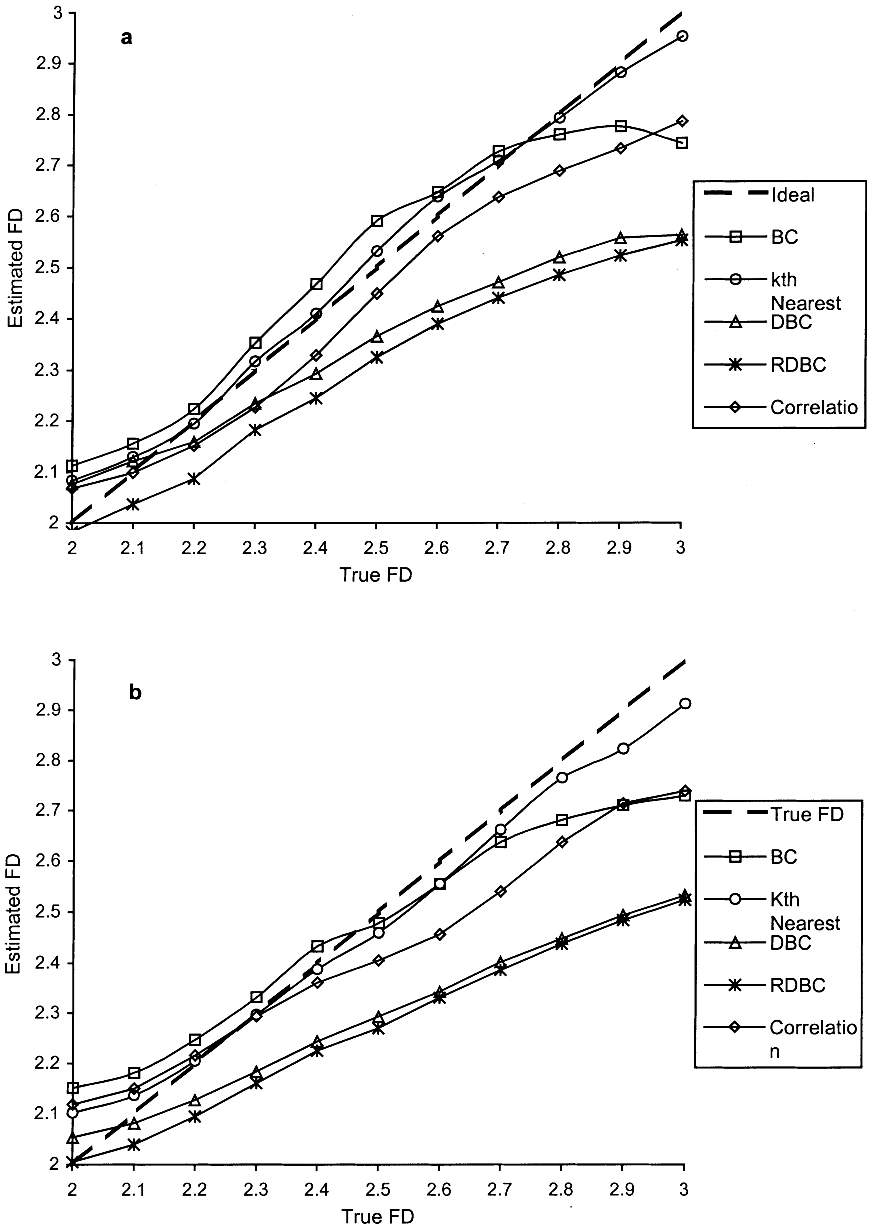

Fig. 2. Estimated fractal dimension obtained by the application of the five methods versus true fractal dimension for images generated (a) by the RMD method and (b) the Fourier Filtering method. Ideal behaviour is indicated by the diagonal, dashed line.

distances, those corresponding to the $k_{\min}$ up to $k_{\max}$ nearest neighbours are recorded as r_{k_m} $(m = 1, 2, \ldots, N_{\text{ref}})$.

Step 4. For $n = 1, 2, \ldots$, the following recursive relations are applied:

$$\gamma_n = D(\gamma_{n-1})/a_{n-1}, \tag{5}$$

$$G(k, \gamma_n) = \langle r_k^{\gamma_{n-1}} \rangle / k^{\gamma_{n-1}/D(\gamma_{n-1})}, \tag{6}$$

where

$$\langle r_k^{\gamma} \rangle = \frac{1}{N_{\text{ref}}} \sum_{m=1}^{N_{\text{ref}}} r_{k_m}^{\gamma}$$

and a_{n-1} is the slope of the best fitting line at the points $(\log k,\ \log \langle r_k^{\gamma_{n-1}} \rangle)$.

Practically, 1000 reference points and two or three iterations (Step 4) are sufficient for a fast and accurate estimation of the fractal dimension. More details of the way that these parameters affect the performance of the algorithm can be found in Section 5.

4. Experimental results

The method is tested on images with known fractal dimension generated by the RMD method (set I) and the Fourier Filtering method (Saupe, 1988, p. 108) (set II). For each set of images and for each value of $\text{FD} = 2.0, 2.1, 2.2, \ldots, 3.0$, a series of 50 images of size 256×256 with 256 grey levels are generated. The number of reference points and iterations are set equal to 1000 and 2, respectively, and the initial value of γ is chosen equal to 2.5. The range of the values of k, over which the slope of the best fitting line is computed, is [50, 250]. The results from the application of the method are compared with those obtained by the box-counting algorithm due to Voss the DBC approach, the RDBC method and the correlation algorithm. For the box-counting algorithm, the number of reference points is set equal to 1500 and the fractal dimension is estimated by the slope of the best fitting line at the points $(-\log r, \log\ N(r))$ for $r = 9, 11, 13, \ldots, 41$. For the DBC approach, the fractal dimension is obtained using boxes of size $r = 5, 6, \ldots, 20$ and for the RDBC method for $r = 7, 8, \ldots, 16$. Finally for the correlation algorithm, the distances between 5 and 25 are used.

The results obtained by the application of the five methods are shown in Fig. 2, where the ideal behaviour (estimated FD = true FD) is illustrated by the diagonal, dashed line. The results indicate that for the two sets of images, the proposed method performs significantly better than the other methods for the full range of variation of fractal dimension. In particular, the proposed method does not underestimate the true value for high fractal dimensions, giving more accurate estimates for the whole range of the fractal dimension.

For the evaluation of the results, the deviation (D) of the estimated (EFD) from the true (TFD) fractal dimension for the two set of images is computed for all the $n = 11$ values of FD by

$$D = \frac{\sqrt{\sum_{i=1}^{n} (\text{EFD}_i - \text{TFD}_i)^2}}{n}.$$

The results, shown in Table 1, highlight the superiority in terms of accuracy of the proposed method against the other methods.

5. Discussion

Three parameters affect the performance of the kth nearest neighbour algorithm. These are: (1) the number of reference points N_{ref}, (2) the number of iterations and (3) the initial value of γ, γ_0.

In Fig. 3, the dependence of the estimated fractal dimension upon the number of reference points, N_{ref}, is illustrated, for the previously generated images by the RMD method for $\text{FD} = 2.0, 2.5$

Table 1
Deviation (D) of the estimated FD from the true value for the two sets of images

Method	D	
	Set I	Set II
Box-counting	0.0313	0.0368
kth nearest neighbour	0.0106	0.0162
DBC	0.0645	0.0789
RDBC	0.0728	0.0823
Correlation	0.0306	0.0409

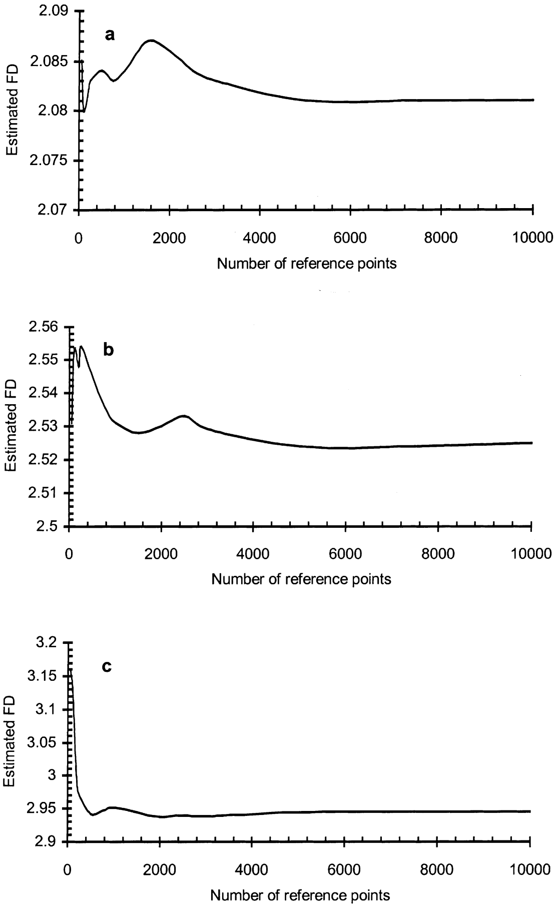

Fig. 3. Estimated fractal dimension versus the number of reference points for (a) FD = 2.0, (b) 2.5 and (c) 3.0, for two iterations with $\gamma_0 = 2.5$.

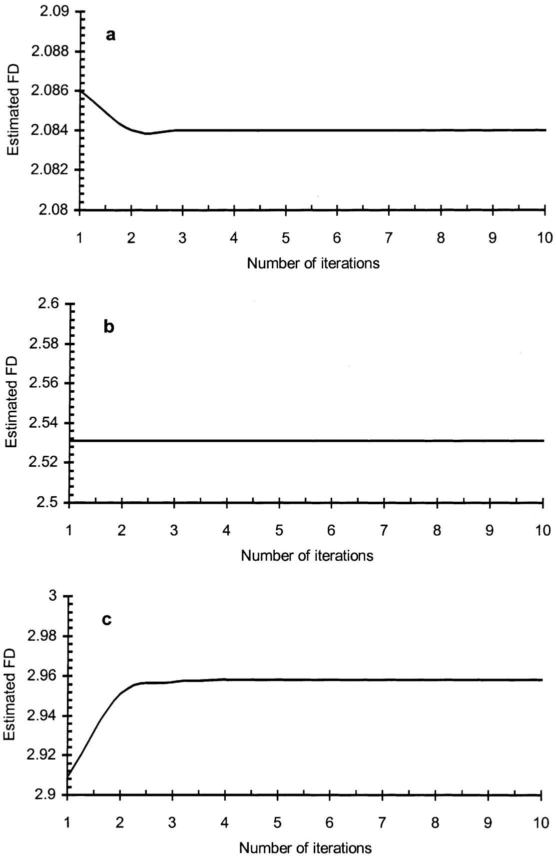

Fig. 4. Estimated fractal dimension versus the number of iterations for (a) FD = 2.0, (b) 2.5 and (c) 3.0, for 1000 reference points and with $\gamma_0 = 2.5$.

and 3.0. The number of iterations is set equal to 2 and the initial value of γ is 2.5. As can be noticed the estimated fractal dimension is almost stable when the number of reference points is greater or equal to 1000. This is the reason for choosing the number of reference points to be equal to 1000 at the experimental results in Section 4.

In Fig. 4, the estimated fractal dimension is plotted versus the number of iterations, for the same set of images. The number of reference points is kept fixed and equal to 1000 and γ_0 is chosen equal to 2.5. The results in Fig. 4 indicate that two iterations suffice in order to obtain an accurate estimation of the true fractal dimension of an image.

Finally, in Fig. 5 the diagram of the estimated fractal dimension versus the initial value of γ is shown. The number of reference points and iterations are set equal to 1000 and 2, respectively (solid line). As can be observed, for $\gamma_0 > 3$ the estimation of the fractal dimension is affected significantly by the initial value of γ, especially for $\mathrm{FD} = 3.0$. However, when the number of iterations is increased to six (dashed line) the estimated fractal dimension is no longer affected by variations of γ_0.

Consequently, accurate estimates of the fractal dimension can be obtained when the number of reference points is, at least, equal to 1000, with two iterations and for an initial value of γ between 2 and 3.

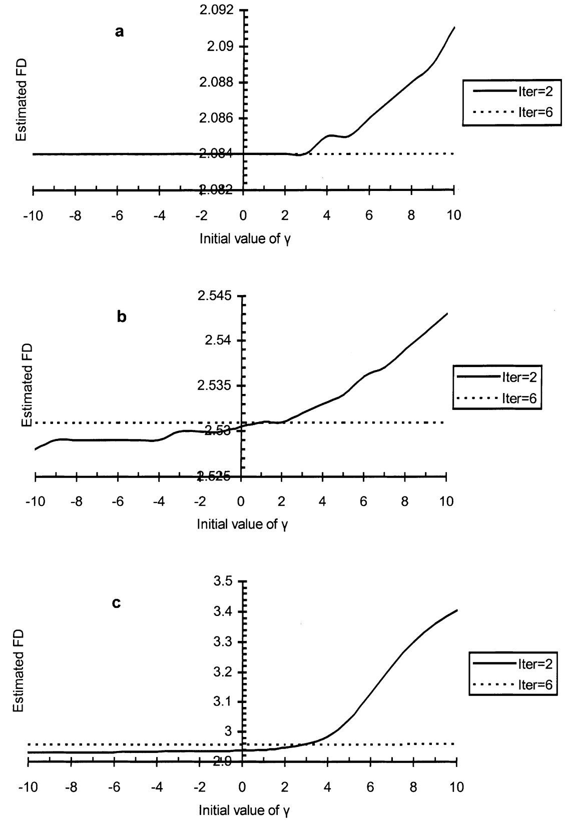

Fig. 5. Estimated fractal dimension versus the initial value of γ for (a) FD = 2.0, (b) 2.5 and (c) 3.0, for 1000 reference points and after two iterations (solid line) and six iterations (dashed line).

A major advantage of the kth nearest neighbour method is its very wide dynamic range regarding the number of points $(\log k,\ \log\langle r_k^{\gamma_{n-1}}\rangle)$ that can be used for the estimation of the fractal dimension. Specifically, for an image of size $N \times N$, the dynamic range of the available points is $O(N^2)$, since for each point there are N^2-1 neighbours. On the other hand, for the box-counting method, the maximum size of cubes that can be used is restricted by the size of the image, which means that, for an $N \times N$ image, the dynamic range is $O(N)$.

Another advantage of the proposed method against box-counting is the significantly less time required for the estimation of the fractal dimension. This superiority is highlighted by the results in Table 2, where the mean value of the processor time (a 200 MHz Pentium PC with 32 MB RAM) required for the estimation of the fractal dimension of images (generated by the RMD method) of different sizes and dimensions is listed. Specifically, for the proposed method and for a given number of reference points and iterations, the required time is reduced significantly as the size of the image increases. This is due to the reduction of the execution time of Step 3, as a result of the increased resolution of the images. Furthermore, the required time is reduced as the fractal dimension gets closer to 2.0 because of the reduced irregularity of the image's intensity surface. This fact implies that the number of iterations in Step 3, for finding the nearest neighbours of the reference points, is much smaller. On the other hand, for the box-counting algorithm, which performed the best after the kth nearest neighbour method, the required time depends only upon the number of reference points and not upon the size or the fractal dimension of the image.

Table 2
Mean value of the processor time required for the estimation of the FD of images with size 128×128, 256×256 and 512×512 (generated by the RMD method), by the application of the kth nearest neighbour and box-counting algorithms

True FD	T_{mean} (s)					
	128×128		256×256		512×512	
	kth nearest	Box-counting	kth nearest	Box-counting	kth nearest	Box-counting
2.0	4.1	13.8	2.7	13.8	1.9	13.7
2.2	4.9	13.9	2.9	13.9	2.1	13.8
2.4	6.0	13.8	4.1	13.8	2.8	13.7
2.6	7.2	13.8	5.4	13.8	3.9	13.8
2.8	8.0	13.7	6.8	13.8	5.7	13.8
3.0	9.5	13.8	8.1	13.8	6.8	13.9

6. Conclusion

In this paper, a method from the field of chaotic dynamics, the kth nearest neighbour, was applied for the estimation of the fractal dimension of grey level images. The method was compared with other well-known estimators of the fractal dimension of images. The methods were tested on computer generated images with known fractal dimension, using two different generation methods (Random Midpoint Displacement and the Fourier Filtering methods). The proposed method was proven superior of the rest algorithms, in terms of accuracy, dynamic range and computational time, for the whole range of fractal dimensions.

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